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K3 曲面のミラーシンメトリー (Mirror symmetry on K3 surfaces)

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Most of the contents are in our paper [Kb2], which we refer for the details.

1. Mirror symmetry

First we introduce some background stories around mirror symmetry. We recommend [M], which is a good introduction to this subject appeared at an early stage.

In the world of $N=2$ superconformal field theories (SCFT), each $N=2$ SCFT has several actions of $N=2$ superconformal algebra (SCA) which have some symmetry called mirror symmetry as described in [LVW] and [AL].

Some of $N=2$ SCFT come from Calabi-Yau manifolds, that is Kähler manifolds whose canonical bundle is trivial. There are at least two ways to construct a topological field theory (TFT) on Calabi-Yau manifolds from a given $N=2$ SCFT. Witten[W] tentatively called those TFT's as A-model and B-model. Those two TFT's are related in the level of representation of $N=2$ SCA by the inversion of the charge and some change of parameter called spectral flow.

In A-model, path-integrals sum up all the possible maps of various degrees with respect to the fixed Kähler class and various mapping degrees from $\mathbb{P}^1$. Thus the quantum product, that is the quantum multiplication structure on the cohomology group of the Calabi-Yau manifold, is, at least, a formal sum of the terms depending on the cohomology class of the image of $\mathbb{P}^1$. Here one can read the 'number of rational curves' with a given degree from an expansion of the quantum product. The resulting cohomology ring, the quantum cohomology ring, is constructed by Ruan and Tian[RT] or by Kontsevich[Kn]. Still, it is open, for instance, whether (1) the formal sum diverges or not; (2) the 'number of rational curves' really count something; the construction of Ruan-Tian uses a generic almost complex structure, which sometimes one can take a complex structure.

On the other hand, in B-model, path-integrals vanish for non-constant maps from $\mathbb{P}^1$ and reduce to some classical explicit period integrals.

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The product is easier to calculate than using A-model. Candelas et al. [CdOGP] uses B-model expansion at a boundary of a family of crepant resolutions of the quotient by a diagonal action of finite group which keeps a holomorphic 3-form on a one-dimensional deformation family of the Fermat quintic hypersurface in $\mathbb{P}^4$, to get the 'numbers' of rational curves of fixed degrees on a generic quintic hypersurface in $\mathbb{P}^4$.

In the following we restrict ourselves in the case that the Calabi-Yau manifold is a K3 surface, namely a compact complex surface whose irregularity is zero and canonical bundle is trivial. Recall that all the K3 surfaces are Kähler.

Let us fix a K3 surface X and let J be a Kähler class on X. J is a real $(1,1)$-class and positive for all the effective curves on X. Take another real $(1,1)$-class B. We call the complex $(1,1)$-class $\omega := B + iJ$ a complexified Kähler class, after Aspinwall-Morrison[AM]. If $\omega^2 \in \mathbb{R}$, then $\omega^2 < 0$ by index theorem. This convention multiplying i to the Kähler class has several numerical naturalities. Remark also that in the dimension-one case, ω sits in the upper half plane and the period, too; the so-called mirror map exchanges those two.

Let A-period $\mho = a(1 \oplus \omega \oplus (-\omega^2/2))$ be an element of $H_A := H^0(\mathbb{C}) \oplus H^{1,1} \oplus H^4(\mathbb{C})$. We heard this definition from Todorov. $\mho$ is of course orthogonal to a period Ω and its complex conjugate $\bar{\Omega}$. One has $\mho \cdot \mho = \bar{\mho} \cdot \bar{\mho} = 0$, $\mho \cdot \bar{\mho} > 0$.

The class $[\mho] \in \mathbb{P}(H_A)$ satisfies the same condition as that of the class of the period, and has one-to-one correspondence to the complexified Kähler class ω.

A rumor says the mirror map μ for the moduli of K3 surfaces with complexified Kähler class can be constructed as follows, possibly by Aspinwall-Morrison and Todorov. Let (X, ω) be a pair of K3 surface and a complexified Kähler class on it. This gives classes of period $[\Omega_X]$ and A-period $[\mho_X]$ in the projectivisation of the total cohomology group $\mathbb{H}^*(X, \mathbb{C})$ which is isometric to $E_8(-1)^{\oplus 2} \oplus U^{\oplus 4}$. If one interchanges $[\Omega]$ and $[\mho]$, H_A and H^2 (I cheated here), uses the surjectivity of the period (and A-period), and gets another pair $(Y, m) = \mu(X, \omega)$. Thus, μ interchanges the complex structure and the complexified Kähler structure. Dolgachev's work[D] is related to the case ω is a generic algebraic class.

By index theorem, we get

proposition. *Assume the mirror map μ is constructed as above. Then* *one has an isometry* $\mu^* : H^*(Y,\mathbb{Z}) \to H^*(X,\mathbb{Z})$ *such that* $\mu^*_{\mathbb{C}}(\mho_Y) = \Omega_X$, $\mu^*_{\mathbb{C}}(\Omega_Y) = \mho_X$. *There is only a finite possible way to take* μ^*, *and* μ^* *is unique on* $(\Omega^\perp \cap \mho^\perp)^\perp$.

I refer also [Kb4] for related topics.

2. Toric mirror symmetry

First recall the original Greene-Plesser construction of mirror manifolds using Gepner models. One takes a defining equation of an n-dimensional Calabi-Yau manifold in a product of weighted projective spaces, which is Fermat-type or almost so. Then the mirror manifold is constructed by taking a (crepant resolution of) quotient by the maximal diagonal projective group which keeps a nonzero holomorphic n-form.

Roan[R] rigorously proved the following prediction in the case of Fermat-type hypersurface in weighted projective 4-spaces : the Euler number of the new threefold differs exactly (-1) times than that of the old threefold.

Batyrev[B] proposed more general construction using 'reflexive' Newton polytopes of defining equations of hypersurfaces in toric Fano varieties, which is generalized by Borisov[Bo] and Batyrev-Borisov[BB1] to the complete intersections. In their construction, the mirror symmetry is simply to take the polar dual polytope. Batyrev showed that the $h^{1,1}$ and $h^{2,1}$ of toric mirror Calabi-Yau threefolds interchange. This result generalizes Roan's result above. In dimension n greater than 3, In fact, Batyrev proved the similar property for $h^{1,1}$ and $h^{n-1,1}$ of the maximal crepant embedded projective partial resolutions of the hypersurfaces in toric Fano varieties. Batyrev-Dais[BD] and Batyrev-Borisov[BB2] defined a 'string-theoretical' Hodge numbers and showed those numbers interchange for the dual polytope, as expected in mirror symmetry.

In this section we only treat the hypersurfaces.

Let M be a free $\mathbb{Z}$-module of rank $(n+1)$ and N be its dual. Let Δ be a $(n+1)$-dimensional strongly convex polytope in $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$; this is equivalent to say Δ is a convex hull of a finite set of points in $M_{\mathbb{Q}}$ which are not on a same hyperplane. One can associate a polarized variety $(\mathbb{P}_\Delta, \mathcal{O}(1))$ to Δ in a natural manner.

Assume that Δ is an integral polytope, that is, all the vertices of Δ is in M. Assume also that $\mathrm{Int}(\Delta) \cap M = \{0\}$ where Int designates the relative interior. Let Δ^* be the polar dual polytope of Δ, namely the subset of $N_{\mathbb{Q}}$

which consists of all the points which is greater than or equal to (-1) on Δ.

If Δ^* is also integral, Δ is said to be *reflexive*. In such a case, the dualizing sheaf of $\mathbb{P}_\Delta$ is invertible and antiample, and general anticanonical divisors $Z(\Delta)$ have only canonical Gorenstein singularities.

The toric mirror symmetry is, for a reflexive polytope Δ, a mirror manifold of a crepant embedded resolution of $Z(\Delta)$ with a general restriction of a complexified Kähler class in the ambient space is to be a similar pair for Δ^*.

Let us think about the case $n = 2$. In this case, the minimal resolution of $Z(\Delta)$ is a K3 surface.

Let L be a free $\mathbb{Z}$-module and take α in $L \otimes \mathbb{C}$. We denote by $\langle \alpha \rangle$ the minimal primitive lattice of L whose $\mathbb{C}$-tensor contains α. See [Kb4] for more details.

$\mathrm{rk}\langle \Omega \rangle$, $\mathrm{rk}\langle \mho \rangle$ and $\langle \Omega, \mho \rangle^\perp$ are computable easily and combinatorially. As for the last two, the lattice structure with respect to the cup product is also easily seen from 1-skeltons of Δ and Δ^*, which is omitted here.

Proposition. *For a reflexive polyhedron Δ,*

(1) $\mathrm{rk}\langle \Omega \rangle = l(\Delta^{[1]}) - 1,$
(2) $\mathrm{rk}\langle \mho \rangle = l(\Delta^{*[1]}) - 1,$
(3) $\mathrm{rk}\langle \Omega, \mho \rangle^\perp = \sum_{\dim \Gamma = 1} l^*(\Gamma) l^*(\Gamma^*).$

Here, l and l^* designate the number of integral point in the polytope and in the relative interior of the polytope, respectively; [1] represents for 1-skelton; Γ moves the set of 1-faces of Δ and Γ^* is the dual 1-face of Γ.

The third lattice in the proposition above is a fixed or rather indeterminacy locus of mirror map on the cohomology groups, which appeared in the first section.

There is a relation between mirror symmetry and singularity theory: the so-called Arnold's strange duality is said to be a prototype of mirror symmetry. We refer [Kb1] for some background and related references.

Theorem. *For each class of fourteen exceptional unimodal critical points, there exists a reflexive polyhedron satisfying the following conditions:*

(1) *the minimal resolution of a weighted hypersurface whose Newton polyhedron is Δ is a K3 surface which is the minimal resolution of*

a non-degenerate hypersurface $Z(\Delta)$, and is a smooth compactification of a Milnor fibre of the original singularity,

(2) *the polar dual polyhedron Δ^* corresponds to the strange dual class.*

(3) $\Omega^\perp \cap \mho^\perp = 0$.

For example, Take $x^8 + y^3 + z^2 = 0$, which is called E_{14} in the list of Arnold et al.[AGV]. The naïve choice of Δ is the full Newton polytope in $\mathbb{P}(1,3,8,12)$ which is the convex hull of the vertices X^8, Y^3, Z^2, W^{24}, that we write here as $\langle X^8, Y^3, Z^2, W^{24}\rangle$, is too large and does not satisfy condition (3). In fact, one should choose $\langle X^4Z, Y^3, Z^2, \dots\rangle$, which defines the isomorphic singularity $x^4z + y^3 + z^2 = 0$. Still, if one take naïvely $\langle X^4Z, Y^3, Z^2, W^{24}\rangle$, which is too small in this case. In fact, one should take as Δ, for example, $\langle X^4Z, Y^3, Z^2, W^6X^6, W^{24}\rangle$. Its dual Newton polyhedron is $\langle X^4, Y^3, XZ^2, W^6Z^2, W^{24}\rangle$ in $\mathbb{P}(1,6,8,9)$ corresponding the singularity of type Q_{10}.

We cite here from [Kb2] the all the Δ's for the exceptional unimodal singularities. For dual singularities, these reflexive polytopes are dual to each other. For self-dual singularities, these examples satisfy $\Delta^* \cong \Delta$. These choices of Δ are not unique in general.

class	a_1 a_2 a_3	h	Δ
E_{12}	6 14 21	42	W^{42}, X^7, Y^3, Z^2
E_{13}	4 10 15	30	W^{30}, W^6X^6, X^5Y, Y^3, Z^2
Z_{11}	6 8 15	30	W^{30}, W^6Y^3, X^5, XY^3, Z^2
E_{14}	3 8 12	24	W^{24}, W^6X^6, X^4Z, Y^3, Z^2
Q_{10}	6 8 9	24	W^{24}, W^6Z^2, X^4, Y^3, XZ^2
Z_{12}	4 6 11	22	W^{22}, W^6X^4, W^4Y^3, X^4Y, XY^3, Z^2
W_{12}	4 5 10	20	W^{20}, $W^{10}Y^2$, $W^2X^2Y^2$, X^5, Y^2Z, Z^2
Z_{13}	3 5 9	18	W^{18}, W^6X^4, W^3Y^3, X^3Z, XY^3, Z^2
Q_{11}	4 6 7	18	W^{18}, W^6X^3, W^4Z^2, X^3Y, Y^3, XZ^2
W_{13}	3 4 8	16	W^{16}, W^4X^4, W^4Y^3, X^4Y, Y^4, Z^2
S_{11}	4 5 6	16	W^{16}, W^6Y^2, W^4Z^2, X^4, XZ^2, Y^2Z
Q_{12}	3 5 6	15	W^{15}, W^6X^3, W^3Z^2, X^3Z, XZ^2, Y^3
S_{12}	3 4 5	13	W^{13}, W^4X^3, W^3Z^2, WY^3, X^3Y, XZ^2, Y^2Z
U_{12}	3 4 4	12	W^{12}, W^4Y^2, W^4Z^2, X^4, Y^2Z, YZ^2

3. Duality of weights

In this section, we translate the polar duality to the language of weights for the special case of weighted projective hypersurfaces which is a compactification of some affine varieties. I refer also [Kb3] for the contents of this and next section.

First we fix some notation.

Let us fix a positive interger N. Let $a = (a_0, a_1, \dots, a_N)$ a $(N+1)$-tuple of positive integers and h be the sum of a_i's.

Take the following subset M_a of $\mathbb{Z}^{N+1}$: $M_a = \{(m_0, \dots, m_N) | \sum_i a_i m_i = h\}$, which we regard as a $\mathbb{Z}$-module by taking $(1, \dots, 1)$ as the origin. M_a is a free $\mathbb{Z}$-module of rank N. We denote its dual by N_a. Let $P_{i,a}$ be the point $(0, \dots, 0, h/m_i, 0, \dots, 0)$ in $M_a \otimes \mathbb{Q}$. We denote the convex hull of $\{\dots, P_{i-1}, P_{i+1}, \dots\}$ by $\pi_{i,a}$ and that of $\{P_0, \dots, P_N\}$ by Δ_a.

Then the associated polarized varity $\mathbb{P}_{\Delta_a}$ is isomorphic to the weighted projective variety $\mathbb{P}(a) = \mathbb{P}(a_0, \dots, a_N)$ by $\mathcal{O}_{\mathbb{P}(a)}(h)$ to $\mathcal{O}_{\mathbb{P}_{\Delta_a}}(1)$.

Let $X(\Delta)$ be a nondegenerate hypersurface whose Newton polytope is Δ.

Proposition. *Let a, b be $(N+1)$-tuple of positive integers. Assume that there exists a $\mathbb{Z}$-isomorphism $\sigma : N_b \to M_a$ such that Δ_a contains $\sigma_{\mathbb{Q}}(\Delta_b^*)$. Then for any reflexive polytope Δ between Δ_a and Δ_b^*, one has a birational map $\mathbb{P}(a)$ to $\mathbb{P}_\Delta$ which induces birational maps α from nondegenerate hypersurfaces $X(\Delta)$ of $\mathbb{P}(a)$ whose newton polytope is Δ, to general anticanonical divisors $Z(\Delta)$ of $\mathbb{P}_\Delta$. Moreover, α are morphisms if $N = 3$.*

Proposition. *Assume $a_0 = b_0 = 1$ and $\sum a_i = \sum b_i = h$. The following two conditions are equivalent.*

(1) *There exists an isomorphism σ as in the theorem above satisfying $\sigma_{\mathbb{Q}}(P_{0,b}^*) \subset \pi_{0,a}$ and $\sigma(\pi_{0,b}^*) = P_{0,a}$.*
(2) *There exists a N-dimensional matrix C whose entry in nonnegative integers satisfying* $C \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} = {}^tC \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix} = \begin{pmatrix} h \\ \vdots \\ h \end{pmatrix}$ *and* $|\det C| = h$.

We say that a and b are dual with respect to C. We call C as a weighted magic square.

As an example, $(1,\dots,1)$ is self-dual, because of the following weighted magic square:

$$C = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{pmatrix}.$$

This corresponds to the fact that one can have a mirror correspondence within hypersurfaces of projective spaces. We will return to this example in the next section.

Theorem. *Let $a = (1, a_1, a_2, a_3)$ be a set of weight appearing in fourteen exceptional unimodal singularities. Then there exists a set of weight $b = (1, b_1, b_2, b_3)$ which is dual with respect to some weighted magic square. Moreover, this b is unique modulo permutation of b_i's and coincides to the weight of strange duality.*

4. Examples and applications

First think about a self-dual weight $(1,1,1,1)$. This weight corresponds to degree four hypersurfaces in $\mathbb{P}^3$. For the sake of simplicity, we will take here as Δ a cone over the vertex W^h.

The biggest subpolyhedron is $\Delta_{(1,1,1,1)}$ itself, which corresponds to the general family of K3 surfaces whose defining equations contain X^h, Y^h, Z^h and W^h. The Picard number of the generic quartic K3 surface is one, and $\Delta^*_{(1,1,1,1)}$ can be embedded in $\Delta_{(1,1,1,1)}$ as

$$\langle W^4, X^2YZ, XY^2Z, XYZ^2\rangle,$$

which corresponds to a smooth compactication S of smoothing of union of four planes in $\mathbb{P}^3$. The Picard number of S is 19.

You may hope there is a 'self-mirror' quartic K3 surface, whose Picard number is half of $h^{1,1}$. In fact, one can easily get a polyhedron

$$\Delta_0 := \langle W^4, Y^2Z^2, Z^2X^2, X^2Y^2\rangle,$$

which satisfies $\Delta_0^* \cong \Delta_0$. The Picard number of the corresponding K3 surface is, however, not 10, but 13. In fact, this polyhedron has nontrivial

$\Omega^\perp \cap \mho^\perp$ whose rank is six. Thus, $h^{1,1} = 20 = 7 + 6 + 7$ and these two 7's are exchanged. In this sense, this is also 'self-mirror'.

If you take another polyhedron such as

$$\Delta_1 := \langle W^4, X^4, X^2Y^2, X^2Z^2, XY^2Z, XYZ^2 \rangle,$$

$\Omega^\perp \cap \mho^\perp$ is zero and the Picard numbers of both families corresponding to Δ_1 or Δ_1^* are ten.

Next think about weighted cases. Some of minimal elliptic weight systems[S], including those of Arnold's strange duality, have their dual weights. These weights have similar property as in strange duality. For instance, $(1,2,3,6)$ and $(1,2,4,5)$ are dual. In this case, a K3 surface which is a compactification of a Milnor fiber of singularity of type $W_{1,0} : x^6 + y^4 + z^2 = 0$, is 'mirror' to that of some nonisolated singularity with $\mathbb{C}^*$-action of weight $(2,4,5)$.

$(1,2,3,4)$ is self-dual and one can construct a series Δ_i's which has strict inclusions $\Delta_1 \supset \cdots \supset \Delta_6$ and satisfies $\Delta_i^* \cong \Delta_{7-i}$.

In our construction, we only treat the hypersurface singularity but which is not necessarily isolated and work on K3 surfaces, rather than on isolated singularities themselves. We note that same Milnor fibres may have several realization as a weighted hypersurface K3 surface or a weighted complete intersection.

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